Theorems of the Propositional Calculus

Equivalence and true

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$

(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$

(3.3) **Axiom, Identity of $\equiv$:** $true \equiv q \equiv q$

(3.4) *true*

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

Negation, inequivalence, and false

(3.8) **Axiom, Definition of** *false***:** $false \equiv \neg true$

(3.9) **Axiom, Distributivity of $\neg$ over $\equiv$:** $\neg(p \equiv q) \equiv \neg p \equiv q$

(3.10) **Axiom, Definition of $\not\equiv$:** $(p \not\equiv q) \equiv \neg(p \equiv q)$

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of** *false***:** $\neg false \equiv true$

(3.14) $(p \not\equiv q) \equiv \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv false$

(3.16) **Symmetry of $\not\equiv$:** $(p \not\equiv q) \equiv (q \not\equiv p)$

(3.17) **Associativity of $\not\equiv$:** $((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \equiv (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \equiv p \equiv q \not\equiv r$

Disjunction

(3.24) **Axiom, Symmetry of $\vee$:** $p \vee q \equiv q \vee p$

(3.25) **Axiom, Associativity of $\vee$:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) **Axiom, Idempotency of $\vee$:** $p \vee p \equiv p$

(3.27) **Axiom, Distributivity of $\vee$ over $\equiv$:** $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$

(3.28) **Axiom, Excluded Middle:** $p \vee \neg p$

(3.29) **Zero of $\vee$:** $p \vee true \equiv true$

(3.30) **Identity of $\vee$:** $p \vee false \equiv p$

(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Conjunction

(3.35) **Axiom, Golden rule:** $p \wedge q \equiv p \equiv q \equiv p \vee q$

(3.36) **Symmetry of $\wedge$:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of $\wedge$:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of $\wedge$:** $p \wedge p \equiv p$

(3.39) **Identity of $\wedge$:** $p \wedge true \equiv p$

(3.40) **Zero of $\wedge$:** $p \wedge false \equiv false$

(3.41) **Distributivity of $\wedge$ over $\wedge$:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(3.42) **Contradiction:** $p \wedge \neg p \equiv false$

(3.43) **Absorption:** (a) $p \wedge (p \vee q) \equiv p$
(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:** (a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of $\wedge$ over $\vee$:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan:** (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) $p \wedge q \equiv p \wedge \neg q \equiv \neg p$

(3.49) $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$

(3.50) $p \wedge (q \equiv p) \equiv p \wedge q$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$

(3.52) **Definition of $\equiv$:** $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Exclusive or:** $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

(3.55) $(p \wedge q) \wedge r \equiv p \equiv q \equiv r \equiv p \vee q \equiv q \vee r \equiv r \vee p \equiv p \vee q \vee r$

Implication

(3.57) **Axiom, Definition of Implication:** $p \Rightarrow q \equiv p \vee q \equiv q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \equiv q \Rightarrow p$

(3.59) **Definition of implication:** $p \Rightarrow q \equiv \neg p \vee q$

(3.60) **Definition of implication:** $p \Rightarrow q \equiv p \wedge q \equiv p$

(3.61) **Contrapositive:** $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

(3.62) $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$

(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:** $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$

(3.64) $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \equiv p$

(3.68) $p \vee (p \Rightarrow q) \equiv true$

(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

(3.70) $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$

(3.71) **Reflexivity of $\Rightarrow$:** $p \Rightarrow p \equiv true$

(3.72) **Right zero of $\Rightarrow$:** $p \Rightarrow true \equiv true$

(3.73) **Left identity of $\Rightarrow$:** $true \Rightarrow p \equiv p$

(3.74) $p \Rightarrow false \equiv \neg p$

(3.75) $false \Rightarrow p \equiv true$

(3.76) **Weakening/strengthening:** (a) $p \Rightarrow p \vee q$
(b) $p \wedge q \Rightarrow p$
(c) $p \wedge q \Rightarrow p \vee q$
(d) $p \vee (q \wedge r) \Rightarrow p \vee q$
(e) $p \wedge q \Rightarrow p \wedge (q \vee r)$

(3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$

(3.78) $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$

(3.79) $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$

(3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \equiv q)$

(3.81) **Antisymmetry:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$

(3.82) **Transitivity:** (a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

Leibniz as an axiom

(3.83) **Axiom, Leibniz:** $e = f \Rightarrow E_e^z = E_f^z$

(3.84) **Substitution:** (a) $(e = f) \wedge E_e^z \equiv (e = f) \wedge E_f^z$
(b) $(e = f) \Rightarrow E_e^z \equiv (e = f) \Rightarrow E_f^z$
(c) $q \wedge (e = f) \Rightarrow E_e^z \equiv q \wedge (e = f) \Rightarrow E_f^z$

(3.85) **Replace by** *true***:** (a) $p \Rightarrow E_p^z \equiv p \Rightarrow E_{true}^z$
(b) $q \wedge p \Rightarrow E_p^z \equiv q \wedge p \Rightarrow E_{true}^z$

(3.86) **Replace by** *false***:** (a) $E_p^z \Rightarrow p \equiv E_{false}^z \Rightarrow p$
(b) $E_p^z \Rightarrow p \vee q \equiv E_{false}^z \Rightarrow p \vee q$

(3.87) **Replace by** *true***:** $p \wedge E_p^z \equiv p \wedge E_{true}^z$

(3.88) **Replace by** *false***:** $p \vee E_p^z \equiv p \vee E_{false}^z$

(3.89) **Shannon:** $E_p^z \equiv (p \wedge E_{true}^z) \vee (\neg p \wedge E_{false}^z)$

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of $\vee$:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(4.3) **Monotonicity of $\wedge$:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

Proof techniques

(4.4) **Deduction:** To prove $P \Rightarrow Q$, assume P and prove Q.

(4.5) **Case analysis:** If E^z_{true}, E^z_{false} are theorems, then so is E^z_P.

(4.6) **Case analysis:** $(p \lor q \lor r) \land (p \Rightarrow s) \land (q \Rightarrow s) \land (r \Rightarrow s) \;\Rightarrow\; s$

(4.7) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.

(4.9) **Proof by contradiction:** To prove P, prove $\neg P \Rightarrow false$.

(4.12) **Proof by contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$

General Laws of Quantification

For symmetric and associative binary operator $\star$ with identity u.

(8.13) **Axiom, Empty range:** $(\star x \mid false : P) = u$

(8.14) **Axiom, One-point rule:** Provided $\neg occurs('x', 'E')$,
$(\star x \mid x = E : P) = P[x := E]$

(8.15) **Axiom, Distributivity:** Provided each quantification is defined,
$(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$

(8.16) **Axiom, Range split:** Provided $R \land S \equiv false$ and each quantification is defined,
$(\star x \mid R \lor S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.17) **Axiom, Range split:** Provided each quantification is defined,
$(\star x \mid R \lor S : P) \star (\star x \mid R \land S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.18) **Axiom, Range split for idempotent** $\star$**:** Prov. each quant. is defined,
$(\star x \mid R \lor S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.19) **Axiom, Interchange of dummies:** Provided each quantification is defined, $\neg occurs('y', 'R')$, and $\neg occurs('x', 'Q')$,
$(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$

(8.20) **Axiom, Nesting:** Provided $\neg occurs('y', 'R')$,
$(\star x, y \mid R \land Q : P) = (\star x \mid R : (\star y \mid Q : P))$

(8.21) **Axiom, Dummy renaming:** Provided $\neg occurs('y', 'R, P')$,
$(\star x \mid R : P) = (\star y \mid R[x := y] : P[x := y])$

(8.22) **Change of dummy:** Provided $\neg occurs('y', 'R, P')$, and f has an inverse, $(\star x \mid R : P) = (\star y \mid R[x := f.y] : P[x := f.y])$

(8.23) **Split off term:** $(\star i \mid 0 \le i < n+1 : P) = (\star i \mid 0 \le i < n : P) \star P^i_n$

Theorems of the Predicate Calculus

Universal quantification

(9.2) **Axiom, Trading:** $(\forall x \mid R : P) \equiv (\forall x \mid : R \Rightarrow P)$

(9.3) **Trading:**
(a) $(\forall x \mid R : P) \equiv (\forall x \mid : \neg R \lor P)$
(b) $(\forall x \mid R : P) \equiv (\forall x \mid : R \land P \equiv R)$
(c) $(\forall x \mid R : P) \equiv (\forall x \mid : R \lor P \equiv P)$

(9.4) **Trading:**
(a) $(\forall x \mid Q \land R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$
(b) $(\forall x \mid Q \land R : P) \equiv (\forall x \mid Q : \neg R \lor P)$
(c) $(\forall x \mid Q \land R : P) \equiv (\forall x \mid Q : R \land P \equiv R)$
(d) $(\forall x \mid Q \land R : P) \equiv (\forall x \mid Q : R \lor P \equiv P)$

(9.5) **Axiom, Distributivity of** $\lor$ **over** $\forall$**:** Prov. $\neg occurs('x', 'P')$,
$P \lor (\forall x \mid R : Q) \equiv (\forall x \mid R : P \lor Q)$

(9.6) Provided $\neg occurs('x', 'P')$, $(\forall x \mid R : P) \equiv P \lor (\forall x \mid : \neg R)$

(9.7) **Distributivity of** $\land$ **over** $\forall$**:** Provided $\neg occurs('x', 'P')$,
$\neg(\forall x \mid : \neg R) \Rightarrow ((\forall x \mid R : P \land Q) \equiv P \land (\forall x \mid R : Q))$

(9.8) $(\forall x \mid R : true) \equiv true$

(9.9) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\forall x \mid R : P) \equiv (\forall x \mid R : Q))$

(9.10) **Range weakening/strengthening:** $(\forall x \mid Q \lor R : P) \Rightarrow (\forall x \mid Q : P)$

(9.11) **Body weakening/strengthening:** $(\forall x \mid R : P \land Q) \Rightarrow (\forall x \mid R : P)$

(9.12) **Monotonicity of** $\forall$**:**
$(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\forall x \mid R : Q) \Rightarrow (\forall x \mid R : P))$

(9.13) **Instantiation:** $(\forall x \mid : P) \Rightarrow P[x := e]$

(9.16) P is a theorem iff $(\forall x \mid : P)$ is a theorem.

Existential quantification

(9.17) **Axiom, Generalized De Morgan:**
$(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$

(9.18) **Generalized De Morgan:**
(a) $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$
(b) $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$
(c) $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \land P)$

(9.20) **Trading:** $(\exists x \mid Q \land R : P) \equiv (\exists x \mid Q : R \land P)$

(9.21) **Distributivity of** $\land$ **over** $\exists$**:** Provided $\neg occurs('x', 'P')$,
$P \land (\exists x \mid R : Q) \equiv (\exists x \mid R : P \land Q)$

(9.22) Provided $\neg occurs('x', 'P')$, $(\exists x \mid R : P) \equiv P \land (\exists x \mid : R)$

(9.23) **Distributivity of** $\lor$ **over** $\exists$**:** Provided $\neg occurs('x', 'P')$,
$(\exists x \mid : R) \Rightarrow ((\exists x \mid R : P \lor Q) \equiv P \lor (\exists x \mid R : Q))$

(9.24) $(\exists x \mid R : false) \equiv false$

(9.25) **Range weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid Q \lor R : P)$

(9.26) **Body weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid R : P \lor Q)$

(9.27) **Monotonicity of** $\exists$**:**
$(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\exists x \mid R : Q) \Rightarrow (\exists x \mid R : P))$

(9.28) **$\exists$-Introduction:** $P[x := E] \Rightarrow (\exists x \mid : P)$

(9.29) **Interchange of quantifications:**
Provided $\neg occurs('y', 'R')$ and $\neg occurs('x', 'Q')$,
$(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$

(9.30) Provided $\neg occurs('\hat{x}', 'Q')$,
$(\exists x \mid R : P) \Rightarrow Q$ is a theorem iff $(R \land P)[x := \hat{x}] \Rightarrow Q$ is a theorem